Lecture 3 Ehresmann connections

In the previous lecture we introduced principal G-bundles : $G \to P \xrightarrow{\pi} M$ is a fibre bundle whose fibres are G-torsors & whose local trivialisations $\pi^{-1}U \to U \times G$ are G-equivariant.

Let $P \xrightarrow{\pi} M$ be a principal G-bundle. Then let $p \in P$ and consider the derivative $(\pi_*)_p : T_pP \to T_{\pi(p)}M$, which is a surjective linear map. Its kernel $V_p := \ker(\pi_*)_p$ is called the vertical subspace. A vector field $\xi \in \mathfrak{X}(P)$ is vertical if $\xi(p) \in V_p$ $\forall p \in P$. The Lie bracket of two vertical vector fields is vertical. The vertical subspaces span a G-invariant integrable distribution $V \subset P$. Indeed, since $\pi \circ r_g = \pi$, $\pi_* \circ (r_g)_* = \pi_*$ and hence $(r_g)_* V_p = V_{pg}$.

11. Definition An Ehresmann connection on P is a smooth choice of horizontal subspaces $H_p \subset T_pP$ complementary to V_p : $T_pP = H_p \oplus V_p$ and such that $(r_g)_* H_p = H_{pg}$. In other words, an Ehresmann connection is a G-invariant distribution $H \subset TP$ complementary to V.

12. Example A G-invariant riemannian metric on P defines an Ehresmann connection by $H_p = V_p^\perp$. This is a familiar construction in Kaluza–Klein theory.

Let $P \xrightarrow{\pi} M$ be a ppal G-bundle. The G action on P defines a smooth map $\mathfrak{g} \to \mathfrak{X}(P)$ assigning to every $X \in \mathfrak{g}$ the fundamental vector field $\xi_X \in \mathfrak{X}(P)$ whose value at p is given by

$$\xi_X(p) = \frac{d}{dt}\left(p e^{tX}\right)\Big|_{t=0}$$

13. Lemma ξ_X is vertical. Proof $\pi_* \xi_X|_p = \frac{d}{dt} \pi(p e^{tX})\big|_{t=0} = \frac{d}{dt}\pi(p)\big|_{t=0} = 0$. ■

Moreover, since the G-action is free, the map $X \mapsto \xi_X(p)$ is an isomorphism $\mathfrak{g} \xrightarrow{\cong} V_p$ $\forall p \in P$.

14. Lemma $(r_g)_* \xi_X = \xi_{Ad_{g^{-1}}(X)}$ for all $g \in G$, $X \in \mathfrak{g}$.

Proof Let $p \in P$. Then,

$$(r_g)_* \xi_X(p) = \frac{d}{dt} r_g(pe^{tX})\Big|_{t=0} = \frac{d}{dt}(pe^{tX}g)\Big|_{t=0} = \frac{d}{dt}(pgg^{-1}e^{tX}g)\Big|_{t=0} = \frac{d}{dt}(pg\, e^{t Ad_{g^{-1}}X})\Big|_{t=0} = \xi_{Ad_{g^{-1}}X}(pg). \quad \blacksquare$$

15. Definition The **connection one-form** of a connection $H \subset TP$ is the $\mathfrak{g}$-valued one-form $\omega \in \Omega^1(P;\mathfrak{g})$ defined by

$$\omega(\xi) = \begin{cases} X & \text{if } \xi = \xi_X \\ 0 & \text{if } \xi \text{ is horizontal} \end{cases}$$

NB: This does define ω by $C^\infty(P)$-linearity.

16. Proposition The connection one-form obeys $r_g^* \omega = Ad_{g^{-1}} \circ \omega$.

Proof Let ξ be horizontal. Then $(r_g^*\omega)(\xi) = \omega((r_g)_*\xi)$ but H is G-invariant & hence $(r_g)_*\xi$ is also horizontal so that $(r_g^*\omega)(\xi) = 0$. But then $Ad_{g^{-1}} \circ \omega(\xi) = 0$ as well. Now let $\xi = \xi_X$ $\exists X \in \mathfrak{g}$. $Ad_{g^{-1}} \circ \omega(\xi_X) = Ad_{g^{-1}}(X)$ but by Lemma 14 $(r_g^*\omega)(\xi_X) = \omega((r_g)_*\xi_X) = \omega(\xi_{Ad_{g^{-1}}X}) = Ad_{g^{-1}}(X)$. $\blacksquare$

Conversely, if $\omega \in \Omega^1(P;\mathfrak{g})$ obeys $r_g^*\omega = Ad_{g^{-1}} \circ \omega$ and $\omega(\xi_X) = X$, then $H := \ker \omega$ is a connection on P.

In Physics we are used to gauge fields defining connections on gauge bundles. Gauge fields are local descriptions of connection one-forms, but they live on M. Recall that we have local sections $s_\alpha : U_\alpha \to \pi^{-1}(U_\alpha)$ associated canonically to the trivialisation: $(\varphi_\alpha \circ s_\alpha)(a) = (a, e)$. We may then pull back ω via s_α: $A_\alpha := s_\alpha^* \omega \in \Omega^1(U_\alpha; \mathfrak{g})$.

17. Proposition Let $\omega_\alpha := Ad_{g_\alpha^{-1}} \circ \pi^* A_\alpha + g_\alpha^* \Theta$ where Θ is the left-invariant Maurer-Cartan form on G. Then $\omega_\alpha = \omega|_{\pi^{-1}U_\alpha}$.

Proof We prove this in two steps:

① ω and ω_α agree on the image of s_α. Since $\pi\circ s_\alpha = id|_{U_\alpha}$, $T_p\mathcal{P} = \mathrm{Im}(s_\alpha\circ\pi)_* \oplus V_p$ for $p = s_\alpha(a)$.
$\therefore$ every $\xi \in T_p\mathcal{P}$ can be written as $\xi = (s_\alpha)_*\pi_*\xi + \xi^v$ $\exists!\ \xi^v \in V_p$. Let's apply ω_α on ξ using that $g_\alpha(p) = e$,

$$\begin{aligned}
\omega_\alpha(\xi) &= (\pi^* s_\alpha^*\omega)(\xi) + (g_\alpha^*\theta_e)(\xi) \\
&= \omega((s_\alpha)_*\pi_*\xi) + \theta_e((g_\alpha)_*\xi) \\
&= \omega((s_\alpha)_*\pi_*\xi) + \theta_e((g_\alpha)_*\xi^v) \qquad \text{since } (g_\alpha)_*(s_\alpha)_* = (g_\alpha\circ s_\alpha)_* = 0 \\
&= \omega((s_\alpha)_*\pi_*\xi) + \omega(\xi^v) \\
&= \omega(\xi)
\end{aligned}$$

② ω and ω_α transform in the same way under the right G-action:

$$\begin{aligned}
r_g^*(\omega_\alpha)_{pg} &= Ad_{g_\alpha(pg)^{-1}} \circ r_g^*\pi^* s_\alpha^*\omega + r_g^* g_\alpha^*\theta \\
&= Ad_{(g_\alpha(p)g)^{-1}} \circ r_g^*\pi^* s_\alpha^*\omega + g_\alpha^* R_g^*\theta \qquad (\text{by } G\text{-equivariance of } g_\alpha) \\
&= Ad_{g^{-1}g_\alpha(p)^{-1}} \circ \pi^* s_\alpha^*\omega + g_\alpha^*(Ad_{g^{-1}}\circ\theta) \qquad (\text{since } \pi\circ r_g = \pi \ \& \text{ transf}^{\underline{n}} \text{ properties of } \theta) \\
&= Ad_{g^{-1}} \circ \left(Ad_{g_\alpha(p)^{-1}} \circ \pi^* s_\alpha^*\omega + g_\alpha^*\theta\right) \\
&= Ad_{g^{-1}} \circ (\omega_\alpha)_p \qquad \therefore\ \omega \ \& \ \omega_\alpha \text{ agree everywhere on } \pi^{-1}(U_\alpha) \ \blacksquare
\end{aligned}$$

But ω is a global one-form on $\mathcal{P}$, hence $\omega_\alpha = \omega_\beta$ on $\pi^{-1}(U_{\alpha\beta})$. This allows us to relate A_α & A_β. Indeed, on $U_{\alpha\beta}$,

$$\begin{aligned}
A_\alpha = s_\alpha^*\omega_\alpha = s_\alpha^*\omega_\beta &= s_\alpha^*\left(Ad_{g_\beta(s_\alpha)^{-1}}\circ\pi^* A_\beta + g_\beta^*\theta\right) \\
&= Ad_{g_{\alpha\beta}} \circ A_\beta + g_{\beta\alpha}^*\theta \qquad (\text{using that } g_\beta\circ s_\alpha = g_\beta\circ g_\alpha^{-1}\circ g_\alpha\circ s_\alpha = g_{\beta\alpha})
\end{aligned}$$

In the language of matrix Lie groups, $g_{\beta\alpha}^*\theta = g_{\beta\alpha}^{-1}\,dg_{\beta\alpha} = g_{\alpha\beta}\,dg_{\alpha\beta}^{-1} = -dg_{\alpha\beta}\,g_{\alpha\beta}^{-1}$ so that

$$A_\alpha = g_{\alpha\beta}A_\beta g_{\alpha\beta}^{-1} - dg_{\alpha\beta}\,g_{\alpha\beta}^{-1} \qquad (\text{gauge transformation!})$$

Similarly, one can ask how $\{A_\alpha\}$ depends on the choice of local section. If $s'_\alpha : U_\alpha \to \pi^{-1}(U_\alpha)$ is another local section, $s'_\alpha(a) = s_\alpha(a) h_\alpha(a)$ $\exists$ smooth $h_\alpha : U_\alpha \to G$ and then $A'_\alpha = \mathrm{Ad}_{h_\alpha^{-1}} \circ A_\alpha + h_\alpha^* \Theta$ which is again of the form of a gauge transformation (by h_α^{-1}).

<u>Upshot</u> There are three ways to understand connections on a principal G-bundle $P \xrightarrow{\pi} M$:

1. a G-invariant horizontal distribution $H \subset TP$
2. a one-form $\omega \in \Omega^1(P; \mathfrak{g})$ satisfying $\omega(\xi_X) = X$ and $r_g^* \omega = \mathrm{Ad}_{g^{-1}} \circ \omega$
3. a family of one-forms $\{A_\alpha \in \Omega^1(U_\alpha; \mathfrak{g})\}$ satisfying $A_\alpha = \mathrm{Ad}_{g_{\alpha\beta}} \circ A_\beta + g_{\beta\alpha}^* \Theta$ on $U_{\alpha\beta} \neq \emptyset$.

If $P \xrightarrow{\pi} M$ is a ppal G-bundle, G-equivariant bundle <u>diffeos</u>
$$\begin{tikzcd} P \arrow[rr, "\Phi"] \arrow[dr] & & P \arrow[dl] \\ & M & \end{tikzcd}$$
are called **gauge transformations**, and one can ask how an Ehresmann connection transforms.
Let $H \subset TP$ be a G-invariant horizontal distribution. Then let $H^\Phi := \Phi_* H$ be the gauge-transformed distro.

<u>18. Lemma</u> $H^\Phi \subset TP$ is an Ehresmann connection.

<u>Proof</u> $(r_g)_* H^\Phi_{\Phi(p)} = (r_g)_* \Phi_* H_p \underset{\Phi \text{ equiv.}}{=} \Phi_* (r_g)_* H_p \underset{H \text{ inv.}}{=} \Phi_* H_{pg} \underset{\text{def}^{\text{n}} \text{ of } H^\Phi}{=} H^\Phi_{\Phi(pg)} \underset{\Phi \text{ equiv.}}{=} H^\Phi_{\Phi(p)g}$ and H^Φ is complementary to V because $\Phi_* : T_pP \xrightarrow{\cong} T_{\Phi(p)}P$ and Φ_* preserves $V = \ker \pi_*$ because $\pi \circ \Phi = \pi$. ∎

<u>Exercise 2</u> Let Φ be a gauge transformation in a ppal G-bundle $P \xrightarrow{\pi} M$. Let ξ_X denote a fundamental vector field for the G-action on P. Show that ξ_X is gauge invariant (i.e.: $\Phi_* \xi_X = \xi_X$) and show that if ω is the connection one-form for an Ehresmann connection H, then $(\Phi^{-1})^* \omega$ is the connection one-form for H^Φ.

Let $\{A_\alpha\}$ and $\{A_\alpha^\Phi\}$ be the gauge fields corresponding to the Ehresmann connections H and H^Φ. How are they related?
First we need to study the local description of Φ. Since Φ preserves fibres, it makes sense to restrict to $\pi^{-1} U_\alpha$.
Applying the trivialisation $\varphi_\alpha(\Phi(p)) = (\pi(p), g_\alpha(\Phi(p)))$ which defines $\phi_\alpha : \pi^{-1} U_\alpha \to G$ by $\phi_\alpha(p) = g_\alpha(\Phi(p)) g_\alpha(p)^{-1}$

The first observation is that $\overline{\phi}_\alpha$ is constant on the fibres:

$$\overline{\phi}_\alpha(pg) = g_\alpha(\Phi(pg))\, g_\alpha(pg)^{-1} \underset{\substack{\uparrow \\ \Phi \text{ equiv.}}}{=} g_\alpha(\Phi(p)g)\, g_\alpha(pg)^{-1} \underset{\substack{\uparrow \\ g_\alpha \text{ equiv.}}}{=} g_\alpha(\Phi(p))\, g\, (g_\alpha(p)g)^{-1} = g_\alpha(\Phi(p))g_\alpha(p) = \overline{\phi}_\alpha(p)$$

and hence it defines a smooth map $\phi_\alpha : U_\alpha \to G$. On overlaps $U_{\alpha\beta} \neq \phi$, we have for all $a \in U_{\alpha\beta}$ and $p \in \pi^{-1}(a)$,

$$\begin{aligned}
\phi_\alpha(a) &= g_\alpha(\Phi(p))\, g_\alpha(p)^{-1} \\
&= g_\alpha(\Phi(p)) \times \underbrace{g_\beta(\Phi(p))^{-1} g_\beta(\Phi(p))}_{e} \times \underbrace{g_\beta(p)^{-1} g_\beta(p)}_{e} \times g_\alpha(p)^{-1} \\
&= g_{\alpha\beta}(a)\; \phi_\beta(a)\; g_{\alpha\beta}(a)^{-1} \qquad \text{since } \pi(p) = \pi(\Phi(p)) = a.
\end{aligned}$$

We will see later that $\{\phi_\alpha\}$ defines a section of a fibre bundle $\mathrm{Ad}\,P$ on M associated to the ppal bundle P.

Exercise 3 Show that on U_α, $A_\alpha^{\Phi} = \mathrm{Ad}_{\phi_\alpha} \circ (A_\alpha - \phi_\alpha^* \theta) \overset{\text{matrix gps}}{=} \phi_\alpha A_\alpha \phi_\alpha^{-1} - d\phi_\alpha \phi_\alpha^{-1}$,
which is indeed a gauge transformation.

These gauge transformations are conceptually different than the ones relating A_α and A_β on $U_{\alpha\beta}$. These ones are "global" objects (sections of $\mathrm{Ad}\,P$) whereas the ones in overlaps are locally defined $\{g_{\alpha\beta}\}$ on non-empty overlaps.